Inviscid Burgers Equation, Painlevé Analysis and a Bäcklund Transformation

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The inviscid Burgers equation is studied with the Painlevé analysis. A Bäcklund transformation is constructed. Then we give the symmetry generators. A two-dimensional case is also investigated.

The particular Monge-Ampére equation

$$\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} + \sigma\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

known as Burgers equation, represents the simplest equation which incorporates both amplitude and diffusion effects. The inviscid Burgers equation is given by

$$\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x}, \tag{2}$$

i.e. we set $\sigma = 0$ in the Burgers equation. Thus the diffusion effects are neglected. It is well known that the inviscid Burgers equation admits the implicit solution [1]

$$u(x,t) = f(x + t\,u(x,t)), \tag{3}$$

where f is a smooth function. From (3) we have

$$\frac{\partial u}{\partial t} = \left(u + t\frac{\partial u}{\partial t}\right) f' = \frac{u}{1 - t f'} f', \tag{4a}$$

$$\frac{\partial u}{\partial x} = \left(1 + t\frac{\partial u}{\partial x}\right) f' = \frac{1}{1 - t f'} f', \tag{4b}$$

whence (2) follows. Obviously we would call (2) integrable. The solution exists as long as $1 - t f' \neq 0$. Hence we can expect that for some x there will be a first t value such that $1 - t f' = 0$. At this point there will be a shock. In transport processes this corresponds to a critical length.

Let us now apply the Painlevé test to (2). The Painlevé test has been described in detail in literature (compare [2, 3, 4] and references therein).

The Painlevé property for ordinary differential equations, namely that all movable singularities in the solutions are poles, has been known since the famous work of Sonya Kovalevskaya, to be related to integrability. In recent years, it has been generalized so as to apply to (nonlinear) partial differential equations. The conjecture is that a partial differential equation is integrable (by the inverse scattering transform or Bäcklund transform, or some other method), if and only if it possesses the Painlevé property (possibly after a change of variables) (see [2, 3, 4] for more details).

The formulation is as follows: Suppose we have a nonlinear partial differential equation with analytic coefficients (or a set of N partial differential equations for N functions, denoted collectively by u). Complexify the functions, so that we now have N holomorphic functions of the complex independent variables. Let ϕ be a holomorphic function such that $\phi = 0$ is not a characteristic hypersurface for the partial differential equation. Let us consider the powers series

$$u = \phi^{-n} \sum_{j=0}^{\infty} u_j \phi^j \tag{5}$$

with n being a suitable positive integer. Now we substitute this power series into the nonlinear partial differential equation. This leads to recursion relations between the u_n. The Painlevé property states that these recursion relations should be consistent, and that the expansion (5) should contain the maximal number of arbitrary functions (counting ϕ as one of them). We also say that the partial differential equation passes the Painlevé test. In the so-called weak Painlevé test we allow n and j to be rational numbers in the expansion (5) (see [2] for more details).

In the present case we find from the Painlevé test that (2) admits an expansion of the form

$$u = u_0 + u_1\phi^n + u_2\phi^{2n} + \ldots, \tag{6}$$

where

$$n = 1/2, \tag{7}$$

and ϕ and u_j are smooth functions of x and t. For the expansion coefficients u_j we obtain the relations collecting terms with the same power of ϕ. We find an infinite coupled system for the expansion coefficients u_j. For the first two expansion coefficients, i.e. u_0, u_1, we find

$$\phi^{-1/2}:\ u_0\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial t}, \tag{8a}$$

$$\phi^0:\ \frac{\partial u_0}{\partial t} + u_0\frac{\partial u_0}{\partial x} + \frac{1}{2}u_1^2\frac{\partial \phi}{\partial x} = 0. \tag{8b}$$

If we require that u_0 satisfies the inviscid Burgers equation

$$\frac{\partial u_0}{\partial t} = u_0\frac{\partial u_0}{\partial x}, \tag{9}$$

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then we find from the infinite coupled system for the expansion coefficient that

$$u_1 = u_2 = \cdots = 0. \tag{10}$$

Inserting (8a) into (9) gives

$$\left(\frac{\partial \phi}{\partial x}\right)^2 \frac{\partial^2 \phi}{\partial t^2} + \left(\frac{\partial \phi}{\partial t}\right)^2 \frac{\partial^2 \phi}{\partial x^2} - 2 \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial t} \frac{\partial^2 \phi}{\partial x \partial t} = 0. \tag{11}$$

Thus from (8a) and (9) it follows that

$$u = \frac{(\partial \phi)/(\partial t)}{(\partial \phi)/(\partial x)}, \tag{12}$$

where ϕ satisfies (11) and u satisfies (2). Thus we have a Bäcklund transformation. Equation (11) can be linearized applying the Legendre transformation [1, 2]. Equation (11) also arises in many non-Painlevé equations at the resonance [2].

Under the transformation

$$\bar{t}(x, t) = t, \tag{13a}$$

$$\bar{x}(x, t) = u(x, t), \tag{13b}$$

$$\bar{u}(\bar{x}(x, t), \bar{t}(x, t)) = x, \tag{13 c}$$

(2) takes the form

$$\frac{\partial \bar{u}}{\partial \bar{t}} = -\bar{x}. \tag{14}$$

The general solution of (14) is given by

$$\bar{u}(\bar{x}, \bar{t}) = g(\bar{x}) - \bar{x}\,\bar{t}, \tag{15}$$

where g is an arbitrary smooth function. Inserting the transformation (13) gives

$$t\,u(x, t) + x = g(u(x, t)). \tag{16}$$

On inspection we see that (2) admits the symmetry generators

$$T = \frac{\partial}{\partial t}, \quad X = \frac{\partial}{\partial x}, \quad S = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, \qquad U = -2t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}. \tag{17}$$

The group theoretical reduction of (2) applying the symmetry generator S yields the solution

$$u(x, t) = -x/t. \tag{18}$$

The group theoretical reduction of (2) applying the symmetry generator U gives the solution

$$u(x, t) = -\frac{x}{2t} \pm \frac{1}{2} \sqrt{\frac{x^2}{t^2} - \frac{4C}{t}}, \tag{19}$$

where C is a constant of integration.

The two dimensional case of the inviscid Burgers equation is given by

$$\frac{\partial u}{\partial t} = u \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y}. \tag{20}$$

We find the implicit solution

$$u(x, y, t) = f(x + u(x, y, t)\,t, y + u(x, y, t)\,t). \tag{21}$$

Again we find an expansion of the form

$$u = u_0 + u_1 \phi^{1/2} + u_2 \phi + \ldots, \tag{22}$$

where

$$u_0 = \frac{\dfrac{\partial \phi}{\partial t}}{\dfrac{\partial \phi}{\partial x} + \dfrac{\partial \phi}{\partial y}} \tag{23}$$

and

$$\frac{\partial u_0}{\partial t} - u_0 \frac{\partial u_0}{\partial x} - u_0 \frac{\partial u_0}{\partial y} - \frac{1}{2} u_1^2 \frac{\partial \phi}{\partial x} - \frac{1}{2} u_1^2 \frac{\partial \phi}{\partial y} = 0. \tag{24}$$

Again we can set $u_1 = u_2 = \cdots = 0$. Then the quantity ϕ satisfies the equation

$$\frac{\partial^2 \phi}{\partial t^2} \left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\right)^2 - 2 \frac{\partial^2 \phi}{\partial t \partial x} \frac{\partial \phi}{\partial t} \left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\right) - 2 \frac{\partial^2 \phi}{\partial t \partial y} \frac{\partial \phi}{\partial t} \left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\right) + \left(\frac{\partial \phi}{\partial t}\right)^2 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + 2 \frac{\partial^2 \phi}{\partial x \partial y}\right) = 0, \tag{25}$$

and (24) takes the form

$$\frac{\partial u_0}{\partial t} - u_0 \frac{\partial u_0}{\partial x} - u_0 \frac{\partial u_0}{\partial y} = 0. \tag{26}$$

To summarize: We find that the inviscid Burgers equation (2) does not pass the Painlevé test, whereas the Burgers equation (1) with $\sigma \neq 0$ passes the test. We find an expansion of the form

$$u = \phi^{-1} \sum_{j=0}^{\infty} u_j \phi^j \tag{27}$$

for the Burgers equation, where ϕ and u_2 are arbitrary. Although the inviscid Burgers equation does not pass the Painlevé test (only the weak Painlevé test), we find a Bäcklund transformation for the inviscid Burgers equation. For the Burgers equation (1) the Painlevé test provides an auto-Bäcklund transformation [2] and a Bäcklund transformation (so-called Cole Hopf transformation [1]) to the linear diffusion equation [2].

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